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Generalized Hamiltonian structures for Ermakov systems

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Abstract

We construct Poisson structures for Ermakov systems using the Ermakov invariant as the Hamiltonian. Two classes of Poisson structures are obtained, one of them is degenerate, in which case we derive the Casimir functions. In some situations, the existence of Casimir functions can give rise to superintegrable Ermakov systems. Finally, we characterize the cases where linearization of the equations of motion is possible.

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1. Introduction

Ermakov systems [1–3] have attracted attention due to both their important physical applications and mathematical properties. A central mathematical property of Ermakov systems is the existence of a constant of motion, the Ermakov invariant. The Ermakov invariant allows us to construct a nonlinear superposition law linking the solutions of the equations of motion composing the Ermakov system [4]. Ermakov systems have recently been of interest in diverse scenarios, such as accelerator physics [5], dielectric planar waveguides [6], cosmological models [7, 8], analysis of supersymmetric families of Newtonian free damping modes [9], study of open fermionic systems [10], analysis of the propagation of electromagnetic waves in one-dimensional inhomogeneous media [11], algebraic approach to integrability of nonlinear systems [12], coupled linear oscillators [13], the semiclassical limit of quantum mechanics [14], supersymmetric quantum mechanics [15], computation of geometrical angles and phases for nonlinear systems [16–18], search for Noether [19, 20] and Lie [21, 22] symmetries, the possible linearization of the system [23, 24], extension of the Ermakov system concept [21, 25–27], the search for additional constants of motion [28] and some discretizations of Ermakov systems [29, 30].

The existence of a Hamiltonian or Lagrangian formulation is a central question for any dynamical system. Cerveró and Lejarreta [31] have identified a Hamiltonian subclass of Ermakov systems and used this Hamiltonian formulation as the starting point for the quantization of these systems. Later, Haas and Goedert extended the class of Hamiltonian Ermakov systems by inclusion of frequency functions depending not only on time, but on dynamical variables as well [32]. Both Hamiltonian formulations for Ermakov systems are canonical formulations, for which the Poisson bracket is defined in the conventional way. On the other hand, non-canonical, or generalized Hamiltonians, or Poisson descriptions, have proved to be relevant in such diverse fields like magnetohydrodynamics, kinetic models in plasma physics, biological models, optics, quantum chromodynamics and so on [33]. There is a wide range of possibilities open when a Poisson formulation is available, such as nonlinear stability analysis through the energy-Casimir method, perturbation methods, integrability results and bifurcation properties [33]. Accordingly, in recent years there has been interest in constructing Poisson structures [34–41], mainly for the special case of three-dimensional dynamical systems.

A finite-dimensional dynamical system is said to be a generalized Hamiltonian system when it can be cast in the form

$$\dot{x}^\mu = J^{\mu\nu}\partial_\nu H \qquad \mu = 1, \ldots, N \tag{1}$$

where sum convention is assumed and $\partial_\mu = \partial/\partial x^\mu$. Here, $H = H(x)$ is the Hamiltonian function and $J^{\mu\nu} = J^{\mu\nu}(x)$ is the Poisson matrix for the system. The Poisson matrix must be skew symmetric, $J^{\mu\nu} = -J^{\nu\mu}$. Moreover, it must satisfy the following set of partial differential equations:

$$J^{\mu\nu}\partial_\nu J^{\rho\sigma} + J^{\rho\nu}\partial_\nu J^{\sigma\mu} + J^{\sigma\nu}\partial_\nu J^{\mu\rho} = 0. \tag{2}$$

These equations ensure that the generalized Poisson bracket, defined as

$$\{A, B\} = \partial_\mu A J^{\mu\nu}\partial_\nu B \tag{3}$$

for any functions $A = A(x)$ and $B = B(x)$, satisfies the Jacobi identity

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0. \tag{4}$$

In fact, equations (2) are a necessary and sufficient condition for the bracket (3) to satisfy the Jacobi identity. Hereafter we refer to (2) as the 'Jacobi identities' too. The above-generalized Poisson brackets are endowed with all properties of conventional Poisson brackets, with the advantage of being applicable to more general systems.

From the definition, we can identify the basic building blocks of any Poisson formulation as being the Hamiltonian function and the Poisson matrix. If a time-independent constant of motion is known, the idea is trying to use it as the Hamiltonian function for the system. In other words, we can look at (1) in a reverse way, as a set of equations for some of the components of the Poisson matrix. Then, for given $\dot{x}^\mu$ and H, system (1) is an undetermined linear system for the matrix elements $J^{\mu\nu}$. Besides being skew symmetric, the Poisson matrix must comply with the Jacobi identities (2), which constitute an overdetermined system of partial differential equations for the remaining components $J^{\mu\nu}$ not fixed by (1). This 'deductive schema' for constructing Poisson structures was developed in detail in [41], and applied to three-dimensional dynamical systems such as Lotka–Volterra systems for three interacting populations, the Rabinovich system and the Rikitake dynamo model [40]. The basic proposition of the present study is to put forward the approach of [41] to find new classes of Ermakov systems for which a Hamiltonian formalism is possible. We use the only time-independent constant of motion always available for Ermakov systems, namely the Ermakov invariant, as the Hamiltonian function and check the consequences of this assumption. This idea is partly inspired by the results of [42], where it was shown that $(n+1)$-dimensional extensions of Ermakov systems, when restricted to the unit sphere S^n, can sometimes be viewed as canonical Hamiltonian systems with the Ermakov invariant playing the role of Hamiltonian. Here, however, there is no restriction to any particular submanifold, and we focus on non-canonical descriptions. Finally, note that Ermakov systems are non-autonomous four-dimensional dynamical systems, in contrast to the earlier studies [40, 41] focused on three-dimensional models.

The paper is organized as follows. In section 2, we set the Hamiltonian for Ermakov systems as the Ermakov invariant, and seek a Poisson matrix complying with the Jacobi identities. In this way, we arrive at two classes of Ermakov systems admitting Poisson structures, analysed in detail in section 3. In section 4, we examine the possibility of applying linearization transforms to the resulting Ermakov systems with Poisson character. Section 5 is dedicated to our final remarks.

2. Poisson structures

Classical Ermakov systems are commonly written in the form

$$\ddot{x} + \omega^2 x = \frac{1}{yx^2} f(y/x) \tag{5}$$

$$\ddot{y} + \omega^2 y = \frac{1}{xy^2} g(x/y) \tag{6}$$

where f and g are arbitrary functions of the indicated variables and ω is an arbitrary frequency function. In most applications, ω is restricted to be dependent on time only, but *here this constraint is relaxed*.

For our purposes, polar coordinates $r = (x^2 + y^2)^{1/2}$ and $\theta = \arctan(y/x)$ are more appropriate. Ermakov systems in polar coordinates read

$$\ddot{r} - r\dot{\theta}^2 + \omega^2 r = \frac{1}{r^3} F(\theta) \tag{7}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -\frac{1}{r^3} G(\theta) \tag{8}$$

where F and G, which depend only on the angle variable, are arbitrary functions suitably related to f and g, respectively. The main property of Ermakov systems is that they always possess a constant of motion, the Ermakov invariant,

$$I = \frac{1}{2}(r^2\dot{\theta})^2 + \int^{\theta} G(\lambda)\,\mathrm{d}\lambda. \tag{9}$$

As manifest by (9), the existence of I is not affected by any particular dependence of the dynamical variables on ω. Also, a little thought shows that for frequency functions depending on dynamical variables we can set $F \equiv 0$ in equation (7) without any loss of generality. However, we keep F mainly for easy comparison with previous results on Ermakov systems.

We want to reformulate our Ermakov system (7)–(8) as a generalized Hamiltonian system as in (1), with all indices running from 1 to 4 since the Ermakov system is a system of two second-order ordinary differential equations. The Poisson matrix $J^{\mu\nu}$ is skew symmetric and should satisfy the following system of partial differential equations:

$$J^{\mu 1}\partial_\mu J^{23} + J^{\mu 2}\partial_\mu J^{31} + J^{\mu 3}\partial_\mu J^{12} = 0 \tag{10}$$

$$J^{\mu 1}\partial_\mu J^{24} + J^{\mu 2}\partial_\mu J^{41} + J^{\mu 4}\partial_\mu J^{12} = 0 \tag{11}$$

$$J^{\mu 1}\partial_\mu J^{34} + J^{\mu 3}\partial_\mu J^{41} + J^{\mu 4}\partial_\mu J^{13} = 0 \tag{12}$$

$$J^{\mu 2}\partial_\mu J^{34} + J^{\mu 3}\partial_\mu J^{42} + J^{\mu 4}\partial_\mu J^{23} = 0. \tag{13}$$

If time-independent, the Hamiltonian H is a constant of motion. The only time-independent constant of motion always available for Ermakov systems, no matter the functions F, G and ω, is the Ermakov invariant. Hence it is natural to define

$$H = I \tag{14}$$

and see the consequences.

The choice of coordinates x^μ is a matter of convenience. Here, we choose

$$(x^1, x^2, x^3, x^4) = (r, \theta, u, v) \tag{15}$$

where

$$u = \dot{r} \qquad v = r^2\dot{\theta}. \tag{16}$$

The Ermakov system viewed as a first-order system then reads

$$\dot{r} = u \tag{17}$$
$$\dot{\theta} = v/r^2 \tag{18}$$
$$\dot{u} = -\omega^2 r + (v^2 + F(\theta))/r^3 \tag{19}$$
$$\dot{v} = -G(\theta)/r^2. \tag{20}$$

Using the Ermakov invariant as the Hamiltonian

$$H = \frac{v^2}{2} + \int^\theta G(\lambda)\,\mathrm{d}\lambda \tag{21}$$

there results, from (1) and (17)–(19), that

$$u = J^{12}G(\theta) + J^{14}v \tag{22}$$
$$v/r^2 = J^{24}v \tag{23}$$
$$-\omega^2 r + (v^2 + F(\theta))/r^3 = J^{32}G(\theta) + J^{34}v. \tag{24}$$

Equation (20) follows automatically from $\dot{H} = 0$ and the skew symmetry of $J^{\mu\nu}$ (see [41] for details).

Let us look more closely at the system (22)–(24). Equation (24) can be viewed as the definition of ω, while equation (22) shows that setting $J^{12} = 0$ eliminates $G(\theta)$ from all considerations.

This is a convenient choice, and still leads to a large class of examples. We found, after long calculations, that it is very hard to impose the Jacobi identities when $J^{12} \neq 0$. Thus, in what follows we set $J^{12} = 0$, leaving $G(\theta)$ arbitrary and allowing for more general classes of Ermakov systems.

Summing up results from (22)–(24) and our choice for J^{12}, we obtain

$$J^{12} = 0 \tag{25}$$
$$J^{14} = u/v \tag{26}$$
$$J^{24} = 1/r^2 \tag{27}$$
$$\omega^2 = (v^2 + F(\theta))/r^4 + (J^{23}G(\theta) - J^{34}v)/r. \tag{28}$$

Our goal now is to insert (25)–(27) into Jacobi identities and solve for the remaining components of the Poisson matrix. With the solution, we can know what are the allowable frequencies using (28).

The second Jacobi identity, equation (11), gives

$$J^{23} = \frac{u}{r^2 v}. \tag{29}$$

Inserting this and (25)–(27) into (10), we obtain

$$u\frac{\partial J^{13}}{\partial u} + v\frac{\partial J^{13}}{\partial v} = J^{13} - \frac{u^2}{v^2} \tag{30}$$

with solution

$$J^{13} = \left(\frac{u}{v}\right)^2 + u\psi\left(\frac{u}{v}, r, \theta, t\right). \tag{31}$$

Here, ψ is an arbitrary function of the indicated arguments. Note that we have included time-dependence for extra generality.

Substituting the already calculated components of the Poisson matrix into (13) gives

$$u\frac{\partial J^{34}}{\partial u} + v\frac{\partial J^{34}}{\partial v} = J^{34} + \frac{2uv}{r}\psi \tag{32}$$

whose solution is

$$J^{34} = \frac{2uv}{r}\psi\left(\frac{u}{v}, r, \theta, t\right) + u\varphi\left(\frac{u}{v}, r, \theta, t\right) \tag{33}$$

where φ is an additional arbitrary function of the indicated arguments.

The only Jacobi identity still deserving attention is equation (12), which yields the consistency condition

$$\psi\varphi' - \psi'\varphi = \frac{\partial\psi}{\partial r} + \frac{v}{r^2 u}\frac{\partial\psi}{\partial\theta} - \frac{2}{r}\psi \tag{34}$$

where the prime denotes differentiation with respect to u/v. For $\psi = 0$, the consistency condition (34) is satisfied in an immediate way leaving φ arbitrary. For $\psi \neq 0$, a different class of solutions arises.

These two possibilities and the associated Poisson structures are studied separately.

3. The two classes of solution

3.1. The case $\psi = 0$

For $\psi = 0$, the consistency condition (34) imposes no constraints on the function φ, which remains arbitrary. From the results of the last section, we obtain the following Poisson matrix:

$$J^{\mu\nu} = \begin{pmatrix} 0 & 0 & (u/v)^2 & u/v \\ 0 & 0 & u/(r^2 v) & 1/r^2 \\ -(u/v)^2 & -u/(r^2 v) & 0 & u\varphi \\ -u/v & -1/r^2 & -u\varphi & 0 \end{pmatrix} \tag{35}$$

where, as said previously, $\varphi = \varphi(u/v, r, \theta, t)$. By construction, this is a Poisson matrix. Moreover, it is not of Lie–Poisson, affine-linear or quadratic type, as more usual [33].

The frequency function of the associated Ermakov system follows from (28),

$$\omega^2 = \frac{1}{r^4}(v^2 + F(\theta)) + \frac{u}{r^3 v}G(\theta) - \frac{uv}{r}\varphi(u/v, r, \theta, t). \tag{36}$$

Using the frequency function as defined in (36) and the definitions of u and v in terms of the original polar coordinates, we derive the following Ermakov system:

$$\ddot{r} = -\frac{\dot{r}}{r^4\dot{\theta}}G(\theta) + r^2\dot{r}\dot{\theta}\varphi\left(\frac{\dot{r}}{r^2\dot{\theta}}, r, \theta, t\right) \tag{37}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -\frac{G(\theta)}{r^3}. \tag{38}$$

We see that the function F disappears from all considerations. By construction, (37)–(38) is an Ermakov system having a Poisson formulation, with the Hamiltonian being the Ermakov invariant and the Poisson matrix given by (35). This is an infinite family of Ermakov systems, containing two arbitrary functions, G and φ. Also note that the frequencies given by (36) cannot be functions of time only, necessarily having a dependence on the dynamical variables.

Here, the Poisson structure is degenerate, as can be readily seen by

$$\det(J^{\mu\nu}) = 0. \tag{39}$$

Therefore, we can obtain Casimir functions, that is, functions which Poisson commute with any function defined on phase space. The defining equations for the Casimir functions, denoted by C, are

$$J^{\mu\nu}\partial_\nu C = 0. \tag{40}$$

The existence of non-constant solutions is due to the degenerate character of the Poisson structure.

There are systematic methods [43, 44] for obtaining the Casimirs, but here a direct approach is sufficient. Using the Poisson matrix, we find the following equations for the Casimirs:

$$u\frac{\partial C}{\partial u} + v\frac{\partial C}{\partial v} = 0 \tag{41}$$

$$\frac{1}{r^2}\frac{\partial C}{\partial \theta} + \frac{u}{v}\frac{\partial C}{\partial r} + u\varphi\frac{\partial C}{\partial u} = 0. \tag{42}$$

The first equation of this system shows that C do depend only on the variables $(u/v, r, \theta, t)$. Taking into consideration this information, we transform equation (42) into

$$\frac{1}{r^2}\frac{\partial C}{\partial \theta} + \alpha\frac{\partial C}{\partial r} + \alpha\varphi(\alpha, r, \theta, t)\frac{\partial C}{\partial \alpha} = 0 \tag{43}$$

where $\alpha = u/v$. The solution for (43) strongly depends on the details of the function φ. Note that $G(\theta)$, the extra arbitrary function defining the Ermakov system, does not play any role in the computation of the Casimirs.

An illuminating way to rewrite (43) is found by means of the change of coordinates

$$\bar{r} = 1/r \qquad \bar{\theta} = \theta \qquad \bar{\alpha} = -\alpha. \tag{44}$$

The equation for the Casimirs becomes

$$\frac{\partial C}{\partial \bar{\theta}} + \bar{\alpha}\frac{\partial C}{\partial \bar{r}} + \frac{\bar{\alpha}}{\bar{r}^2}\varphi(-\bar{\alpha}, 1/\bar{r}, \bar{\theta}, t)\frac{\partial C}{\partial \bar{\alpha}} = 0. \tag{45}$$

For $\bar{\theta}$ interpreted as an independent variable, $\bar{r}$ as a coordinate and $\bar{\alpha}$ as a velocity, this is Liouville's equation for the invariants of the equations of motion,

$$\frac{\mathrm{d}\bar{r}}{\mathrm{d}\bar{\theta}} = \bar{\alpha} \qquad \frac{\mathrm{d}\bar{\alpha}}{\mathrm{d}\bar{\theta}} = \frac{\bar{\alpha}}{\bar{r}^2}\varphi(-\bar{\alpha}, 1/\bar{r}, \theta, t) \tag{46}$$

which are also the characteristic equations for (45). Note that here the physical time t is a mere parameter.

Equations (46) are equivalent to the Newton equation for one-dimensional motion under the force field $\bar{\alpha}\varphi/\bar{r}^2$. For functions φ yielding completely integrable examples of such motions, we can find all the Casimirs for the Poisson structure (35). To show a concrete example where this is possible, consider the case

$$\varphi = -\frac{\bar{r}^2}{\bar{\alpha}}\frac{\mathrm{d}V}{\mathrm{d}\bar{r}}(\bar{r}, t) \tag{47}$$

for an arbitrary pseudo-potential $V(\bar{r}, t)$. In this case, Newton's equation that follows from (46) is

$$\frac{\mathrm{d}^2\bar{r}}{\mathrm{d}\bar{\theta}^2} = -\frac{\mathrm{d}V}{\mathrm{d}\bar{r}}(\bar{r}, t) \tag{48}$$

an autonomous potential system, since $\bar{\theta}$ does not appear explicitly. As for any autonomous one-dimensional potential system, there is a complete integrability. The constants of motion are

$$C_1 = \frac{1}{2}\left(\frac{\mathrm{d}\bar{r}}{\mathrm{d}\bar{\theta}}\right)^2 + V(\bar{r}, t) \tag{49}$$

$$C_2 = \bar{\theta} - \frac{1}{\sqrt{2}}\int^{\bar{r}} \frac{\mathrm{d}\lambda}{(C_1 - V(\lambda, t))^{1/2}} \tag{50}$$

respectively the energy and the additional integration constant for the equations of motion. In terms of the original coordinates of the Poisson description, the quantities (49)–(50) are

$$C_1 = \frac{1}{2}\left(\frac{u}{v}\right)^2 + V(1/r, t) \tag{51}$$

$$C_2 = \theta - \frac{1}{\sqrt{2}}\int^{1/r} \frac{\mathrm{d}\lambda}{(C_1 - V(\lambda, t))^{1/2}} \tag{52}$$

which are the Casimirs for the Poisson structure (35) when φ is given as in (47). When V does not contain the time explicitly, the Casimirs become additional constants of motion for the Ermakov system. In this case, we obtain a superintegrable [45] class of Ermakov systems, possessing three invariants, namely the Ermakov invariant and the two Casimir functions. In fact, we can derive superintegrable Ermakov systems in all cases when (45) can be solved in closed form for the two Casimirs of the Poisson structure, and when φ is time-independent.

Finally, let us examine more closely the superintegrable Ermakov systems with the Casimirs C_1 and C_2 given in (51)–(52), in the special situation for which $\partial V/\partial t = 0$. Using C_2 as in (52) and the implicit function theorem, we locally obtain the equation for the orbits, $r = r(\theta, C_1, C_2)$. Now, using the Ermakov invariant, we get the angle as a function of time through the quadrature

$$t + k = \int^{\theta} \frac{r^2(\lambda, C_1, C_2)\,\mathrm{d}\lambda}{h(\lambda, I)} \tag{53}$$

where k is the last integration constant and

$$h(\theta, I) = \sqrt{2}\left(I - \int^{\theta} G(\lambda)\,\mathrm{d}\lambda\right)^{1/2}. \tag{54}$$

Locally, (54) gives θ as a function of time and four integration constants, namely I, C_1, C_2 and k.

To show an example of the procedure, consider the particular case

$$V(\bar{r}) = \frac{1}{2\bar{r}^2} \tag{55}$$

for which (48) describes a singular oscillator. Using (47), the result is

$$\varphi = -\frac{r^3\dot{\theta}}{\dot{r}} \tag{56}$$

and, from (37)–(38), we obtain the following Ermakov system:

$$\ddot{r} = -\frac{\dot{r}}{r^4\dot{\theta}}G(\theta) - r^5\dot{\theta}^2 \tag{57}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -\frac{1}{r^3}G(\theta). \tag{58}$$

We left the function $G(\theta)$ undetermined. From the pseudo-potential (55) and the orbit equation (52), we get

$$r^2 = \frac{2C_1}{1 + 4C_1^2(\theta - C_2)^2} \tag{59}$$

which shows a spiral motion of a particle coming arbitrarily close to the origin. The time-dependence of such motion is obtained from the quadrature (53), which depends on the details of the function $G(\theta)$.

3.2. The case $\psi \neq 0$

For $\psi \neq 0$, the consistency condition (34) has a different class of solutions,

$$\varphi = \left(\int^{u/v} \frac{\mathrm{d}\lambda}{\psi^2(\lambda, r, \theta, t)}\left(\frac{\partial\psi}{\partial r}(\lambda, r, \theta, t) + \frac{1}{r^2\lambda}\frac{\partial\psi}{\partial\theta}(\lambda, r, \theta, t) - \frac{2}{r}\psi(\lambda, r, \theta, t)\right)\right.$$
$$\left. + \chi(r, \theta, t)\right)\psi(u/v, r, \theta, t) \tag{60}$$

where χ is an arbitrary function of the indicated arguments. Therefore, we obtain the Poisson structure

$$J^{\mu\nu} = \begin{pmatrix} 0 & 0 & (u/v)^2 + u\psi & u/v \\ 0 & 0 & u/(r^2v) & 1/r^2 \\ -(u/v)^2 - u\psi & -u/(r^2v) & 0 & u\varphi + 2uv\psi/r \\ -u/v & -1/r^2 & -u\varphi - 2uv\psi/r & 0 \end{pmatrix} \tag{61}$$

with φ specified in terms of χ and ψ according to (60). The Poisson structure is non-degenerate,

$$\det(J^{\mu\nu}) = \frac{u^2\psi}{r^4}\left(\frac{2u}{v^2} + \psi\right) \neq 0. \tag{62}$$

The following frequency functions are derived from (28):

$$\omega^2 = \frac{1}{r^4}(v^2 + F(\theta)) + \frac{u}{r^3v}G(\theta) - \frac{uv}{r}\left(\varphi(u/v, r, \theta, t) + 2\frac{v}{r}\psi(u/v, r, \theta, t)\right) \tag{63}$$

which again necessarily depend on the dynamical variables. The resulting Ermakov systems are

$$\ddot{r} = -\frac{\dot{r}}{r^4\dot{\theta}}G(\theta) + r^2\dot{r}\dot{\theta}\left(\varphi\left(\frac{\dot{r}}{r^2\dot{\theta}}, r, \theta, t\right) + 2r\dot{\theta}\psi\left(\frac{\dot{r}}{r^2\dot{\theta}}, r, \theta, t\right)\right) \tag{64}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -\frac{G(\theta)}{r^3}. \tag{65}$$

These Ermakov systems contain three arbitrary functions, namely G, ψ and χ. Since the Poisson structure is non-degenerate, there are no nontrivial Casimirs. Therefore, we found a new class of Ermakov systems that can be cast in a non-degenerate generalized Hamiltonian form.

4. Linearization

Ermakov systems with frequency functions depending only on time are shown to be linearizable [23] using

$$\bar{r} = 1/r \tag{66}$$

as the in dependent variable and the angle θ as the independent one. This change of variables is accomplished by the relation

$$\dot{\theta} = h(\theta, I)/r^2 \tag{67}$$

where $h(\theta, I)$ is defined in (54). In terms of $\bar{r}$, θ, the Ermakov systems transform into a one-parameter family of second-order linear ordinary differential equations depending on the value of the Ermakov invariant [23], whenever the frequency function does not contain dynamical variables. This result was used for the analysis of the stability and periodicity of some Ermakov systems arising in two-layer, shallow water wave theory [45]. The linearization transform (66)–(67) was shown to be useful also for a class of Ermakov systems for which the frequency function is not a mere function of time [24]. This result provides an explanation why Kepler–Ermakov systems [47], a perturbation of conventional Ermakov systems, are linearizable.

In view of the usefulness of the linearization transform (66)–(67) in several contexts, it is interesting to check if it can also be possible for our classes of Ermakov systems admitting Poisson formulations. For instance, applying (66)–(67) to the Ermakov system (37)–(38) results

$$\frac{\mathrm{d}^2\bar{r}}{\mathrm{d}\theta^2} = \frac{1}{\bar{r}^2}\frac{\mathrm{d}\bar{r}}{\mathrm{d}\theta}\varphi\left(-\frac{\mathrm{d}\bar{r}}{\mathrm{d}\theta}, \frac{1}{\bar{r}}, \theta, t\right). \tag{68}$$

For $\partial\varphi/\partial t \neq 0$, (68) becomes an integro-differential equation, a possibility we will not consider here. Equation (68) is equivalent to equation (46), which determines the Casimirs for the Poisson structure.

The term on the right-hand side of (68) has a linear character if and only if

$$\frac{1}{\bar{r}^2}\frac{\mathrm{d}\bar{r}}{\mathrm{d}\theta}\varphi\left(-\frac{\mathrm{d}\bar{r}}{\mathrm{d}\theta}, \frac{1}{\bar{r}}, \theta, t\right) = A(\theta)\frac{\mathrm{d}\bar{r}}{\mathrm{d}\theta} + B(\theta)\bar{r} + C(\theta) \tag{69}$$

for functions A, B and C depending only on the angle. If the assumption (69) is satisfied, the Ermakov systems (37)–(38) fall in the class of linearizable Ermakov systems discussed in [24]. Moreover, if (69) is valid, the characteristic equations (48) for the Casimirs are also linear. This does not imply, of course, that the Casimirs may always be found in closed form when (69) holds. Similar remarks apply to the linearization of our second class of Ermakov systems admitting a Poisson formulation, treated in section 3.2.

5. Conclusion

In this paper we have proposed the Ermakov invariant as the Hamiltonian function and reformulate the Ermakov system as a Poisson system. The main difficulty is to find a Poisson matrix reproducing the equations of motion and, at the same time, being compatible with the Jacobi identities. However, the task was achieved and two classes of the Poisson structures were derived. One of them is degenerate, thus opening the possibility of constructing Casimir invariants. These Casimirs, if time-independent, are also constants of motion. In the cases where the Casimirs are time-independent and available in closed form, we obtain superintegrable Ermakov systems. A class of such superintegrable Ermakov systems was explicitly shown in section 3.1. Another, non-degenerate, class of Ermakov systems admitting generalized Hamiltonian formulation with the Ermakov invariant playing the role of Hamiltonian function was also found. In this latter case, no Casimirs exist. Both classes of Ermakov systems are specified by two arbitrary functions. All these considerations apply to frequency functions having a dependence on the dynamical variables. Finally, the possibility of linearization of the equations of motion was analysed in section 4. Interestingly, we found that the determining equations for the Casimirs are linear when the associated Ermakov systems are linearizable through (66)–(67).

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